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The Fluctuation Rate of the Chi-Process

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Richard A. Silverman

Abstract

The chi-process is defined as a natural generalization of the chi-distribution of statistical theory. A formula is derived for the expected number of level-crossings per second of the chi-process. The formula contains as a special case the familiar expression for the fluctuation rate of the envelope of Gaussian noise.

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1. Introduction

The expected number of times per second that a stationary random process $\xi(t)$ assumes the value a is given by the familiar expression[1]

$$N(a) = \int_{-\infty}^{\infty} |\dot{\xi}|\, p(a,\dot{\xi})\, d\dot{\xi}, \tag{1}$$

where $p(a,\dot{\xi})$ is the bivariate probability density of ξ and its time derivative $\dot{\xi}$, evaluated at $\xi = a$. Consider first the case where ξ denotes a zero-mean Gaussian process $g(t)$ with the correlation function

$$R(\tau) = \frac{\langle g(t)g(t+\tau)\rangle}{\langle g^2(t)\rangle} = \int_0^{\infty} \Phi(\omega)\cos\omega\tau\, d\omega.$$

Here the angular parentheses denote the expectation or ensemble average, and $\Phi(\omega)$ is the (normalized) power spectrum of the process $g(t)$. Then, equation (1) reduces to[1]

$$N_g(a) = \sqrt{\frac{2}{\pi}}\, \sigma \sqrt{-R_0''}\, W_g(a), \tag{2}$$

where $\sigma^2 = \langle g^2(t)\rangle$, R_0'' is the second derivative of $R(\tau)$ with respect to τ evaluated at $\tau = 0^*$, and

$$W_g(x) = (1/\sqrt{2\pi}\,\sigma)\exp(-x^2/2\sigma^2)$$

is the univariate probability density of the process $g(t)$, i.e., the probability density of a Gaussian random variable with mean zero and variance σ^2. Alternatively, (2) can be written as

$$N_g(a) = \sqrt{\frac{2}{\pi}}\, \sigma \left(\overline{\omega^2}\right)^{1/2} W_g(a), \tag{3}$$

where

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*The prime will be used to denote differentiation with respect to τ, and the dot to denote differentiation with respect to t.

$$\overline{\omega^2} = \int_0^\infty \omega^2 \, \Phi(\omega) d\omega$$

is the mean square (angular) frequency of the process g(t). Consider next the case where ξ denotes the envelope E(t) of the Gaussian process g(t). Then, as shown by several authors[2],[3],[4], equation (1) reduces to

$$(4) \qquad N_E(a) = \sqrt{\frac{2}{\pi}} \, \sigma \, \Delta\omega \, W_2(a),$$

where

$$(5) \qquad \begin{aligned} W_2(x) &= (x/\sigma^2) \exp(-x^2/2\sigma^2), & x \geq 0, \\ W_2(x) &= 0 \qquad\qquad\qquad\qquad , & x < 0 \end{aligned}$$

is the univariate probability density of E(t), i.e., the familiar Rayleigh distribution. The quantity $\Delta\omega$ is given by

$$(6) \qquad (\Delta\omega)^2 = \overline{\omega^2} - (\overline{\omega})^2 ,$$

where $\overline{\omega^2}$ is the mean square frequency of the underlying Gaussian process g(t), and

$$\overline{\omega} = \int_0^\infty \omega \, \Phi(\omega) d\omega$$

is its mean frequency. It follows at once that any two of the quantities $\overline{\omega}$, $W_g(a)$, $N_E(a)$ determine the other.* The similarity of (3) and (4) is apparent.

In this note we define an infinite class of random processes, the so-called chi-processes. We find that the expression for the fluctuation rate of these processes is a natural generalization of (2), (3), and (4).

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*If we regard $\Phi(\omega)$ as a distribution of unit mass on the interval $(0,\infty)$, then $\overline{\omega}$ is the centroid of $\Phi(\omega)$, $\overline{\omega^2}$ is the moment of inertia of $\Phi(\omega)$ about the origin, and $(\Delta\omega)^2$ is the moment of inertia of $\Phi(\omega)$ about its centroid. With this interpretation, the relation between $\overline{\omega}$, $N_g(a)$, and $N_E(a)$ is an instance of the parallel axis theorem of mechanics.

2. The chi-process and its fluctuation rate

We begin by defining the chi-process by generalizing to random processes the familiar definition of the chi-distribution[5].

Definition: Let $g_1(t),\ldots,g_n(t)$ be n independent stationary Gaussian random processes such that

$$\langle g_i(t)\rangle = 0 \;,$$

$$\langle g_i^2(t)\rangle = \sigma^2 \;, \tag{7}$$

$$\langle g_i(t)g_i(t+\tau)\rangle = \sigma^2 R(\tau), \qquad 1 \leq i \leq n.$$

Then by the chi-process (χ-process) with n degrees of freedom is meant the random process

$$\chi_n(t) = \left[\sum_{i=1}^{n} g_i^2(t)\right]^{1/2}. \tag{8}$$

The processes $g_i(t)$, $1 \leq i \leq n$, can be regarded as the components of a vector random process $\vec{g}(t)$, and the process $\chi_n(t)$ as the length of $\vec{g}(t)$. The conditions (7) mean that the components of $\vec{g}(t)$ have mean zero and are identically autocorrelated. The univariate distribution of the process $\chi_n(t)$ is, of course, the χ_n-distribution, with probability density $W_n(x)$ given by[5]

$$W_n(x) = \frac{2}{2^{n/2}\,\sigma^n\,\Gamma\left(\frac{n}{2}\right)}\, x^{n-1} \exp(-x^2/2\sigma^2), \qquad x \geq 0,$$

$$W_n(x) = 0 \qquad\qquad , \qquad x < 0. \tag{9}$$

(Note the consistency of (5) and (9).) We now derive an expression for the fluctuation rate of the chi-process.

Theorem: The expected number of times per second that the process $\chi_n(t)$ assumes the value a is given by

(10) $$N_n(a) = \sqrt{\frac{2}{\pi}}\ \sigma \sqrt{-R_o''}\ W_n(a).$$

Proof. We consider first the trivial but exceptional case n = 1, and note that $\chi_1(t) = \sqrt{g^2(t)} = |g(t)|$. Since $|g(t)| = a \geq 0$ if and only if $g(t) = a$ or $g(t) = -a$, it follows from (2) that

$$N_1(a) = \sqrt{\frac{2}{\pi}}\,\sigma \sqrt{-R_o''}\ 2W_g(a)\ , \qquad a \geq 0\ ,$$

$$N_1(a) = 0 \qquad , \qquad a < 0\ .$$

Since by (9)

$$W_1(a) = 2W_g(a)\ , \qquad a \geq 0,$$

$$W_1(a) = 0 \qquad , \qquad a < 0,$$

equation (10) is established for the case n=1.*

When $n \geq 2$, we must apply the basic formula (1), which requires that we first calculate the bivariate probability density $p(\chi_n, \dot{\chi}_n)$ of χ_n and its derivative $\dot{\chi}_n$. We begin by writing the probability density of the 2n jointly Gaussian random variables $x_1 = g_1, \ldots, x_n = g_n$, $x_{n+1} = \dot{g}_1, \ldots, x_{2n} = \dot{g}_n$. (Since all the processes are stationary, the argument t can be dropped.) It follows from (7) and the independence of the processes $g_i(t)$, $1 \leq i \leq n$, that the matrix M with elements $M_{ij} = \langle x_i x_j \rangle$ is diagonal with non-zero elements

$$M_{11} = \ldots = M_{nn} = \sigma^2 \quad ,$$

$$M_{n+1,n+1} = \ldots = M_{2n,2n} = -\sigma^2 R_o'' \,.$$

The probability density of the 2n random variables $x_1, \ldots, x_{2n}$ is

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[*] Unlike the case $n \geq 2$, $\chi_1(t)$ assumes the value zero (without passing through it) at a non-zero rate. Although the sample functions of $\chi_1(t)$ are not differentiable at their zeros, the latter are a set of measure zero, so that the sample functions are differentiable almost everywhere.

$$p(x_1,\ldots,x_{2n}) = \frac{1}{(2\pi)^n(\det M)^{1/2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2n} M_{ij}^{-1}\, x_i x_j\right] ,$$

where M_{ij}^{-1} are the elements of the reciprocal matrix M^{-1}. Since

$$\det M = \sigma^{4n}(-R_o'')^n ,$$

and M^{-1} is diagonal with non-zero elements

$$M_{11}^{-1} = \ldots = M_{nn}^{-1} = \frac{1}{\sigma^2} ,$$

$$M_{n+1,n+1}^{-1} = \ldots = M_{2n,2n}^{-1} = \frac{1}{-\sigma^2 R_o''} ,$$

we find, returning to the variables g_i and $\dot{g}_i$, that

$$p(g_1,\ldots,g_n,\dot{g}_1,\ldots,\dot{g}_n) = \frac{1}{(2\pi\sigma^2)^n(-R_o'')^{n/2}} \exp\left[-\frac{\sum_{i=1}^{n} g_i^2}{2\sigma^2} - \frac{\sum_{i=1}^{n} \dot{g}_i^2}{-2\sigma^2 R_o''}\right] .$$

We now introduce n-dimensional spherical coordinates by writing

$$g_1 = \chi_n \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-1} ,$$

$$g_2 = \chi_n \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{n-1} ,$$

....

$$g_{n-1} = \chi_n \sin\theta_1 \cos\theta_2 ,$$

$$g_n = \chi_n \cos\theta_1 ,$$

which maps the region $-\infty < g_i < \infty$, $1 \le i \le n$, into the region

$$0 \le \chi_n < \infty,$$

$$0 \leq \theta_1, \ldots, \theta_{n-2} \leq \pi,$$

$$0 \leq \theta_{n-1} < 2\pi.$$

We see at once that

$$\sum_{i=1}^{n} g_i^2 = \chi_n^2 ,$$

and after a bit of manipulation that

$$(11) \quad \sum_{i=1}^{n} \dot{g}_i^2 = \dot{\chi}_n^2 + \chi_n^2 \dot{\theta}_1^2 + \chi_n^2 \dot{\theta}_2^2 \sin^2\theta_1 + \ldots + \chi_n^2 \dot{\theta}_{n-1}^2 \sin^2\theta_1 \ldots \sin^2\theta_{n-2}.$$

Thus, the joint probability density of the new random variables $\chi_n, \theta_1, \ldots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1}$ is

$$p(\chi_n, \theta_1, \ldots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1})$$

$$= \frac{1}{(2\pi\sigma^2)^n(-R_0'')^{n/2}} \exp\left[-\frac{\chi_n^2}{2\sigma^2} - \frac{\dot{\chi}_n^2 + \chi_n^2\dot{\theta}_1^2 + \chi_n^2\dot{\theta}_2^2 \sin^2\theta_1 + \ldots + \chi_n^2\dot{\theta}_{n-1}^2 \sin^2\theta_1 \ldots \sin^2\theta_{n-2}}{-2\sigma^2 R_0''} \right]$$

$$\times \frac{\partial(g_1, g_2, \ldots, g_n, \dot{g}_1, \dot{g}_2, \ldots, \dot{g}_n)}{\partial(\chi_n, \theta_1, \ldots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1})} \quad .$$

It follows from Laplace's expansion theorem for determinants [6], and the fact that the partial derivatives of the variables $g_1, g_2, \ldots, g_n$ with respect to the variables $\dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1}$ vanish, that

$$\frac{\partial(g_1, g_2, \ldots, g_n, \dot{g}_1, \dot{g}_2, \ldots, \dot{g}_n)}{\partial(\chi_n, \theta_1, \ldots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1})} = \frac{\partial(g_1, g_2, \ldots, g_n)}{\partial(\chi_n, \theta_1, \ldots, \theta_{n-1})} \frac{\partial(\dot{g}_1, \dot{g}_2, \ldots, \dot{g}_n)}{\partial(\dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1})}$$

$$= \left[\frac{\partial(g_1, g_2, \ldots, g_n)}{\partial(\chi_n, \theta_1, \ldots, \theta_{n-1})} \right]^2 = J_n^2 ,$$

where the Jacobian

$$J_n = \chi_n^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}$$

is the square root of the product of the coefficients of the quantities $\dot{\chi}_n^2, \dot{\theta}_1^2, \ldots, \dot{\theta}_{n-1}^2$ in the sum (11).

The bivariate probability density $p(\chi_n, \dot{\chi}_n)$ is given by

$$p(\chi_n, \dot{\chi}_n) = \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} \int_{-\infty}^{\infty} d\dot{\theta}_1 \ldots \int_{-\infty}^{\infty} d\dot{\theta}_{n-1}\, p(\chi_n, \theta_1, \ldots, \theta_{n-1}, \dot{\chi}_n, \dot{\theta}_1, \ldots, \dot{\theta}_{n-1})$$

which equals

$$p(\chi_n, \dot{\chi}_n) = \frac{1}{(2\pi\sigma^2)^n (-R_o'')^{n/2}} \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1} J_n^2 \exp\left[-\frac{\chi_n^2}{2\sigma^2} - \frac{\dot{\chi}_n^2}{-2\sigma^2 R_o''}\right]$$

(12)

$$\times \int_{-\infty}^{\infty} d\dot{\theta}_1 \exp\left[-\frac{\chi_n^2 \dot{\theta}_1^2}{-2\sigma^2 R_o''}\right] \int_{-\infty}^{\infty} d\dot{\theta}_2 \exp\left[-\frac{\chi_n^2 \dot{\theta}_2^2 \sin^2\theta_1}{-2\sigma^2 R_o''}\right] \ldots \int_{-\infty}^{\infty} d\dot{\theta}_{n-1} \exp\left[-\frac{\chi_n^2 \dot{\theta}_{n-1}^2 \sin^2\theta_1 \ldots \sin^2\theta_{n-2}}{-2\sigma^2 R_o''}\right].$$

The integrals with respect to $\dot{\theta}_1, \ldots, \dot{\theta}_{n-1}$ can be done immediately. Their product is

$$\left[\frac{-2\pi\sigma^2 R_o''}{\chi_n^2}\right]^{\frac{n-1}{2}} \frac{1}{\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}} = \frac{(-2\pi\sigma^2 R_o'')^{\frac{n-1}{2}}}{J_n} ,$$

so that (12) becomes

$$p(\chi_n, \dot{\chi}_n) = \frac{1}{(2\pi\sigma^2)^{\frac{n+1}{2}} \sqrt{-R_o''}} \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1}\, J_n \exp\left[-\frac{\chi_n^2}{2\sigma^2} - \frac{\dot{\chi}_n^2}{-2\sigma^2 R_o''}\right] .$$

Since

$$\int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} d\theta_{n-1}\, J_n = \chi_n^{n-1} A_n ,$$

where

$$A_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

is the area of the n-dimensional sphere with unit radius, we find, using (9), that

$$p(\chi_n, \dot{\chi}_n) = W_n(\chi_n) \frac{1}{\sqrt{-2\pi R''_o}\,\sigma} \exp\left[- \frac{\dot{\chi}_n^2}{-2\sigma^2 R''_o}\right].$$

It follows that at any time t, the process $\chi_n(t)$ and its derivative $\dot{\chi}_n(t)$ are independent. Indeed, the univariate distribution of $\dot{\chi}_n(t)$ is Gaussian, with mean zero and variance $-\sigma^2 R''_o$, although its integral $\chi_n(t)$ is manifestly non-Gaussian. This involves no contradiction, since the process $\dot{\chi}_n(t)$ is not multivariate Gaussian. The proof of the theorem is completed by noting that

$$N_n(a) = \int_{-\infty}^{\infty} |\dot{\chi}_n| p(a, \dot{\chi}_n) d\dot{\chi}_n = \frac{2W_n(a)}{\sqrt{-2\pi R''_o}\,\sigma} \int_0^{\infty} \dot{\chi}_n \exp\left[- \frac{\dot{\chi}_n^2}{-2\sigma^2 R'_o}\right] d\dot{\chi}_n ,$$

so that

$$(10) \qquad N_n(a) = \sqrt{\frac{2}{\pi}}\, \sigma \sqrt{-R''_o}\, W_n(a), \qquad \text{Q. E. D.}$$

In proving the theorem, the independence of the processes $g_1(t),\ldots,g_n(t)$ was used only to infer that

$$(13) \quad \langle g_i(t) g_j(t) \rangle = \langle g_i(t) \dot{g}_j(t) \rangle = \langle \dot{g}_i(t) \dot{g}_j(t) \rangle = 0, \quad 1 \leq i \neq j \leq n.$$

(The relations $\langle g_i(t) \dot{g}_i(t) \rangle = 0$, $1 \leq i \leq n$, follow automatically from $R'(0) = 0$.) Thus we have the following result.

> *Corollary: Let* $g_1(t),\ldots,g_n(t)$ *be n jointly Gaussian processes which are stationary and stationarily correlated and which satisfy* (7). *Suppose in addition that the cross-correlations of* $g_1(t),\ldots,g_n(t)$ *are such that the conditions* (13) *are satisfied. Then the expected*

number of times per second that the process $\left[\sum_{i=1}^{n} g_i^2(t)\right]^{1/2}$ assumes the value a is identical with that of the χ_n-process, and is given by (10).

3. Application to the envelope fluctuation rate

We now show that the general expression (10) includes as a special case the formula (4) for the fluctuation rate of the envelope E(t) of a zero-mean stationary Gaussian process g(t). First we recall the following facts about the envelope[1].

1) For every $\omega_o > 0$ there is a representation of g(t) of the form

$$g(t) = g_c(t) \cos \omega_o t - g_s(t) \sin \omega_o t,$$

where the processes $g_c(t)$ and $g_s(t)$ are jointly Gaussian, and

$$\langle g_c(t) \rangle = \langle g_s(t) \rangle = 0,$$

$$\langle g_c^2(t) \rangle = \langle g_s^2(t) \rangle = \sigma^2 .$$

2) The correlation properties of $g_c(t)$ and $g_s(t)$ are given by

$$\langle g_c(t) g_c(t+\tau) \rangle = \langle g_s(t) g_s(t+\tau) \rangle = \sigma^2 R(\tau),$$

(14)

$$\langle g_c(t) g_s(t+\tau) \rangle = -\langle g_s(t) g_c(t+\tau) \rangle = \sigma^2 S(\tau),$$

where

$$R(\tau) = \int_0^\infty \Phi(\omega) \cos[(\omega-\omega_o)\tau] d\omega,$$

(15)

$$S(\tau) = \int_0^\infty \Phi(\omega) \sin[(\omega-\omega_o)\tau] d\omega.$$

The quantity

(16) $$k^2(\tau) = R^2(\tau) + S^2(\tau)$$

is independent of ω_o.

3) In particular, it follows from (14) and (15) that*

$$\left\langle g_c(t)g_s(t)\right\rangle = S_o = 0,$$

$$\left\langle \dot{g}_c(t)\dot{g}_s(t)\right\rangle = S_o'' = 0,$$

and that

$$\left\langle g_c(t)\dot{g}_s(t)\right\rangle = -\left\langle \dot{g}_c(t)g_s(t)\right\rangle = S_o' = \int_0^\infty (\omega-\omega_o)\,\Phi(\omega)\,d\omega = \bar{\omega} - \omega_o ,$$

where $\bar{\omega}$ is the centroid of $\Phi(\omega)$, regarded as a mass distribution.

4) The envelope E(t) is defined by

$$E(t) = \left[g_c^2(t) + g_s^2(t) \right]^{\frac{1}{2}} ,$$

and is independent of the choice of ω_o.

It follows that the conditions needed to apply the corollary of the fluctuation rate theorem are met, provided that we choose the frequency ω_o, which is at our disposal, to be the centroid of the distribution $\Phi(\omega)$. With this choice

$$- R_o'' = \int_0^\infty (\omega-\bar{\omega})^2\,\Phi(\omega)\,d\omega = (\Delta\omega)^2 ,$$

where $(\Delta\omega)^2$ is the quantity defined by (6). Thus, the envelope fluctuation rate $N_E(a)$ is given by

$$N_E(a) = N_2(a) = \sqrt{\frac{2}{\pi}}\,\sigma\sqrt{-R_o''}\,W_2(a),$$

or

---

* The subscript zero denotes evaluation at $\tau = 0$.

$$(4) \qquad N_E(a) = \sqrt{\tfrac{2}{\pi}}\,\sigma\Delta\omega\, W_2(a),$$

which completes the proof that (4) is a special case of (10).

It is interesting to note another expression for $N_E(a)$ in terms of the envelope correlation function

$$R_E(\tau) = \frac{\langle E(t)E(t+\tau)\rangle - \langle E\rangle^2}{\langle E^2\rangle - \langle E\rangle^2} .$$

It can be shown that[7]

$$R_E(\tau) = \frac{2E(k)-(1-k^2)K(k)-(\pi/2)}{2-(\pi/2)} ,$$

where K(k) and E(k) are the complete elliptic integrals of the first and second kind, respectively, and k is the function of τ defined by (16). It follows that

$$(17) \qquad -R_E''(0) = \frac{2}{4-\pi}\,(\partial/\partial k)\Big[2E(k)-(1-k^2)K(k)\Big]_{k=1}\,(-R_o'' - S_o'^2) = \frac{2}{4-\pi}\,(\Delta\omega)^2 ,$$

where we have used formulas 112.01, 710.02, and 710.04 from Byrd and Friedman's handbook of elliptic integrals[8]. Comparing (4) and (17), we see that another expression for $N_E(a)$ is

$$N_E(a) = \sqrt{\tfrac{4-\pi}{\pi}}\,\sigma\sqrt{-R_E''(0)}\; W_2(a).$$

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